Principle of Minimum Potential Energy for a Two-Spring Bistable System

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June 18, 2026

1 Introduction

The *Principle of Minimum Potential Energy* is a fundamental concept in structural mechanics. It states that an elastic structure in stable equilibrium adopts a configuration that makes its total potential energy stationary and, for stable equilibrium, locally minimum.

In this note, we use a very simple nonlinear structure to demonstrate the principle. The structure contains two identical axial springs connected at a middle joint. The two outer ends are fixed, and a vertical displacement is applied at the middle joint while the reaction force is recorded. Although each spring is linear in its own axial direction, the force-displacement relation of the whole system is nonlinear because the geometry changes during deformation.

This example is similar to the classical von Mises truss. It is useful for beginners because the whole derivation can be obtained from geometry and energy, without using advanced continuum mechanics.

2 Problem Setup

Consider two identical axial springs connected at the middle point, as shown in Fig. 1. The left and right end points are fixed. The middle joint can move only in the vertical direction.

The geometry and material parameters are:

- Initial spring length: l_0,
- Axial spring stiffness: k,
- Initial angle between each spring and the horizontal direction: θ_0.

The boundary conditions and loading condition are:

- The two outer end points are fixed,
- The middle point is constrained to move only in the vertical direction,
- A vertical displacement u is applied to the middle point,
- The corresponding vertical reaction force is denoted by F.

In this note, the downward displacement and downward reaction force are taken as positive. The initial vertical height of the middle point is

$$h_0 = l_0 \sin\theta_0, \tag{2.1}$$

and the half-span of the structure is

$$a = l_0 \cos\theta_0. \tag{2.2}$$

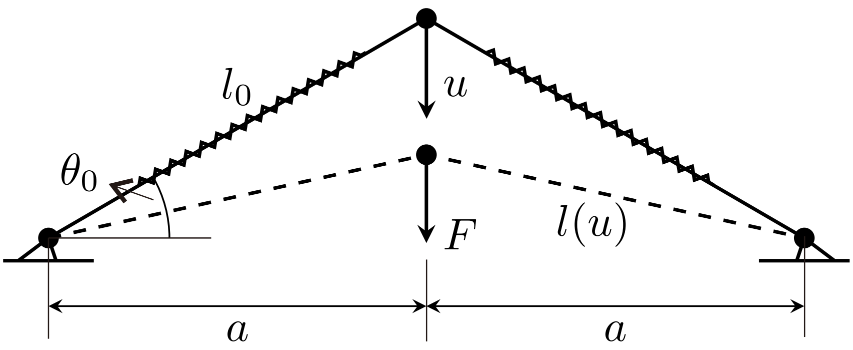

Figure 1: Two axial springs connected at the middle point. The solid lines show the initial configuration, and the dashed lines show a deformed configuration.

3 Geometry of the Deformed Configuration

After the middle point moves downward by u, its vertical distance from the support line becomes

$$h(u) = h_0 - u. \tag{3.1}$$

The current length of each spring is therefore

$$l(u) = \sqrt{a^2 + \left(h_0 - u\right)^2}. \tag{3.2}$$

The axial elongation of each spring is

$$\Delta l = l(u) - l_0. \tag{3.3}$$

It is important to notice that the displacement u is vertical, but the spring deformation is axial. The connection between them is purely geometric. This is the reason why a system made from linear springs can still have a nonlinear force-displacement curve.

4 Total Potential Energy

The total potential energy Π of the system is the strain energy stored in the two springs minus the work of the generalized force conjugate to the middle-point displacement:

$$\Pi = U - W. \tag{4.1}$$

The strain energy stored in one spring is

$$U_{\text{one spring}} = \frac{1}{2}k\big(l - l_0\big)^2. \tag{4.2}$$

Since there are two identical springs, the total strain energy is

$$U = 2 \times \frac{1}{2}k\big(l - l_0\big)^2 = k\big(l - l_0\big)^2. \tag{4.3}$$

For the present one-degree-of-freedom system, the generalized force is the vertical reaction force F required to hold the middle point at displacement u. Its work-conjugate term is

$$W = Fu. \tag{4.4}$$

Thus, the total potential energy can be written as

$$\Pi(u) = k\big[l(u) - l_0\big]^2 - Fu. \tag{4.5}$$

5 Principle of Minimum Potential Energy

The equilibrium configuration is obtained by requiring the first variation of the total potential energy to vanish:

$$\delta\Pi = 0. \tag{5.1}$$

For this system, there is only one unknown displacement u. Therefore, the variation condition is equivalent to

$$\frac{d\Pi}{du} = 0. \tag{5.2}$$

Using Eq. (4.5), we have

$$\frac{d\Pi}{du} = 2k\big(l - l_0\big)\frac{dl}{du} - F. \tag{5.3}$$

The derivative of the current spring length is

$$\frac{dl}{du} = \frac{d}{du}\sqrt{a^2 + \big(h_0 - u\big)^2} = -\frac{h_0 - u}{l}. \tag{5.4}$$

Substituting this result into the stationary condition gives

$$-2k\big(l - l_0\big)\frac{h_0 - u}{l} - F = 0. \tag{5.5}$$

Therefore, the force-displacement relation is

$$F(u) = -2k\big[l(u) - l_0\big]\frac{h_0 - u}{l(u)}. \tag{5.6}$$

Equivalently,

$$F(u) = 2k\big[l_0 - l(u)\big]\frac{h_0 - u}{l(u)}. \tag{5.7}$$

This is the main result obtained from the principle of minimum potential energy.

Stability from the Second Variation

The first variation gives equilibrium, but it does not tell us whether the equilibrium is stable. To check stability, we examine the second variation of the total potential energy. For a system with one generalized displacement, this is equivalent to checking the second derivative

$$\delta^2\Pi = \frac{d^2\Pi}{du^2}(\delta u)^2. \tag{5.8}$$

Because $(\delta u)^2$ is positive for any nonzero virtual displacement, the sign of $d^2\Pi/du^2$ determines the type of equilibrium:

$$\frac{d^2\Pi}{du^2} > 0 \quad \text{stable equilibrium,} \tag{5.9}$$

$$\frac{d^2\Pi}{du^2} < 0 \quad \text{unstable equilibrium.} \tag{5.10}$$

At equilibrium, the generalized force is $F = dU/du$, where U is the strain energy. Therefore,

$$\frac{d^2\Pi}{du^2} = \frac{dF}{du}. \tag{5.11}$$

The limit points satisfy

$$\frac{dF}{du} = 0. \tag{5.12}$$

At these points, the second variation changes sign. This is why the limit points separate stable and unstable parts of the equilibrium path for a force-controlled interpretation.

6 Specific Example, Stability, and Discussion

For the present example,

$$\theta_0 = 30^\circ, \qquad \sin\theta_0 = \frac{1}{2}, \qquad \cos\theta_0 = \frac{\sqrt{3}}{2}. \tag{6.1}$$

Thus,

$$h_0 = \frac{1}{2}l_0, \qquad a = \frac{\sqrt{3}}{2}l_0. \tag{6.2}$$

The current spring length becomes

$$l(u) = \sqrt{\left(\frac{\sqrt{3}}{2}l_0\right)^2 + \left(\frac{1}{2}l_0 - u\right)^2}. \tag{6.3}$$

Substituting this into Eq. (5.6), the dimensional force-displacement relation is

$$F(u) = 2k\left[l_0 - \sqrt{\frac{3}{4}l_0^2 + \left(\frac{1}{2}l_0 - u\right)^2}\right]\frac{\frac{1}{2}l_0 - u}{\sqrt{\frac{3}{4}l_0^2 + \left(\frac{1}{2}l_0 - u\right)^2}}. \tag{6.4}$$

For plotting and interpretation, it is useful to introduce the nondimensional displacement

$$\eta = \frac{u}{l_0}, \tag{6.5}$$

and the nondimensional force

$$P = \frac{F}{kl_0}. \tag{6.6}$$

Then the current nondimensional spring length is

$$\lambda(\eta) = \frac{l}{l_0} = \sqrt{\frac{3}{4} + \left(\frac{1}{2} - \eta\right)^2}. \tag{6.7}$$

The nondimensional force-displacement relation is

$$P(\eta) = 2\big[1 - \lambda(\eta)\big]\frac{\frac{1}{2} - \eta}{\lambda(\eta)}. \tag{6.8}$$

This equation gives the complete equilibrium path of the two-spring system. The corresponding nondimensional strain energy stored in the two springs is

$$\bar{U}(\eta) = \frac{U}{kl_0^2} = \big[\lambda(\eta) - 1\big]^2. \tag{6.9}$$

Figure 2 shows the nondimensional force, the nondimensional strain energy, and the corresponding deformation snapshots for the specific case $\theta_0 = 30°$. The five red points are used to connect the equations with the actual deformation of the structure:

(i) initial state,

(ii) upper limit point,

(iii) neutral zero-force state, which is an unstable equilibrium,

(iv) lower limit point,

(v) inverted zero-force state, which is the second stable state.

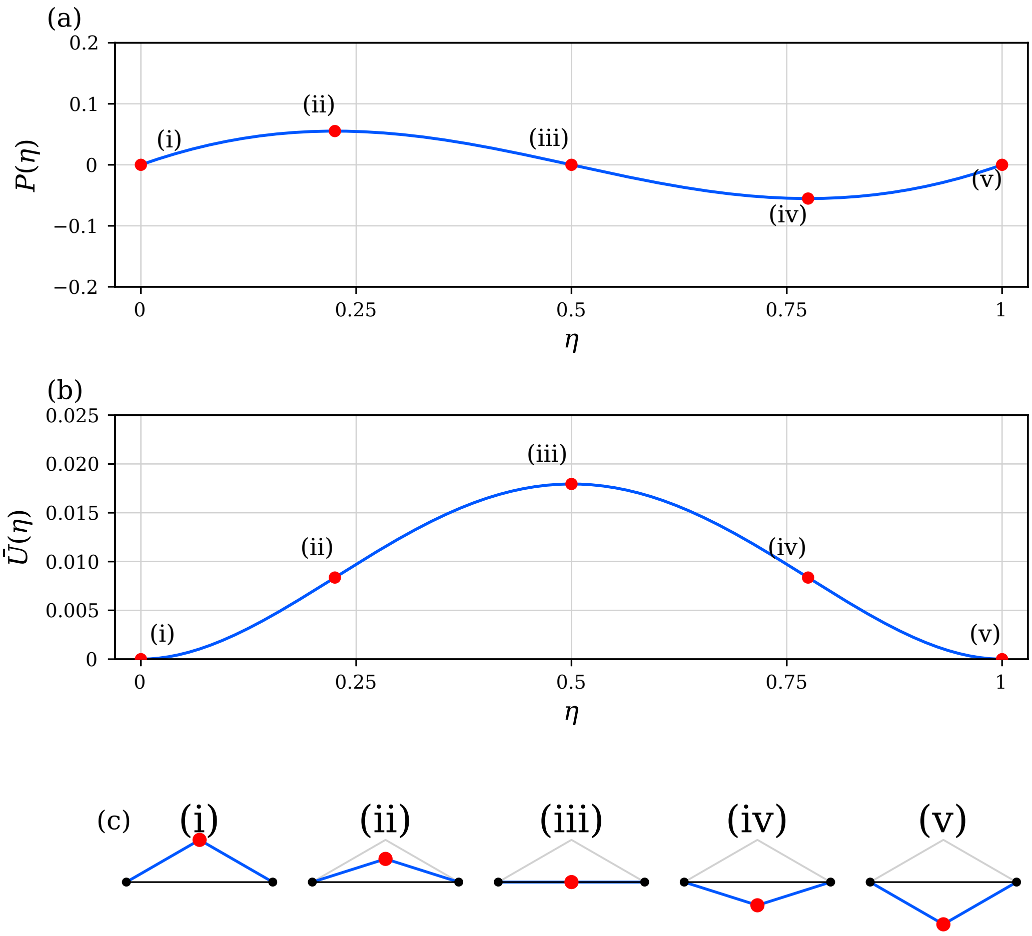

Figure 2: Bistable response for $\theta_0 = 30°$: (a) nondimensional reaction force $P(\eta) = F/(kl_0)$, (b) nondimensional strain energy $\bar{U}(\eta) = U/(kl_0^2)$, and (c) configuration snapshots corresponding to the five red points.

The energy curve helps explain why the system is bistable. At state (i), the springs have their original length and the stored strain energy is zero. This is one stable state. As the joint moves downward, the springs shorten and store elastic energy. At state (ii), the structure reaches the upper limit point. Between states (ii) and (iv), the slope dF/du is negative, so the second variation of the total potential energy is negative. This part of the equilibrium path is unstable for a force-controlled interpretation.

State (iii) is a zero-force equilibrium state because the vertical component of the spring force vanishes when the middle joint is on the support line. However, it is not a stable state: the strain energy is at a local maximum, so a small perturbation moves the system away from this configuration.

After passing the neutral position, the sign of the vertical force changes. State (iv) is the lower limit point, and state (v) is the inverted zero-force configuration where the spring length again equals l_0. This is the second stable state of the system. The two zero-energy configurations, states (i) and (v), are separated by an unstable equilibrium at state (iii). This is the essential feature of a bistable structure: the system has two stable configurations and an energy barrier between them.

7 Summary

In this lecture note, the principle of minimum potential energy was applied to a two-spring structure. The geometry was first described using the middle-point displacement, and the spring elongation was then used to write the total potential energy. By applying the stationary condition of the total potential energy, the nonlinear force-displacement relation was obtained.

The example with $\theta_0 = 30^\circ$ shows that a structure made from linear springs can still have a strongly nonlinear response. The reason is geometric nonlinearity: as the middle point moves, the spring length and the spring direction both change. The force curve, strain-energy curve, and deformation snapshots together show the bistable response, including the upper limit point, the neutral unstable state, the lower limit point, and the inverted stable state. The second variation of the total potential energy provides the stability check: stable branches have positive tangent stiffness, while the middle branch has negative tangent stiffness.